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A note on monotone covering properties ☆

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ABSTRACT

In this note, we show that a monotonically normal space that is monotonically countably metacompact (monotonically meta-Lindelöf) must be hereditarily paracompact. This answers a question of H.R. Bennett, K.P. Hart and D.J. Lutzer. We also show that any compact monotonically meta-Lindelöf T_2-space is first countable. In the last part of the note, we point out that there is a gap in Proposition 3.8 which appears in [H.R. Bennett, K.P. Hart, D.J. Lutzer, A note on monotonically metacompact spaces, Topology Appl. 157 (2) (2010) 456–465]. We finally give a detailed proof of how to overcome the gap.

1. Introduction

In [2], the relation between separability and monotone Lindelöf property in generalized ordered (GO)-spaces was investigated. A topological space X is *monotonically Lindelöf* [2] if for each open cover $\mathcal{U}$ of X there is a countable open cover $r(\mathcal{U})$ of X that refines $\mathcal{U}$ and has the property that if an open cover $\mathcal{U}$ refines an open cover $\mathcal{V}$, then $r(\mathcal{U})$ refines $r(\mathcal{V})$. In this case, r will be called a *monotone Lindelöf operator* for the space X.

For two collections $\mathcal{U}$ and $\mathcal{V}$ of subsets of a space X, we write $\mathcal{U} \prec \mathcal{V}$ to mean that for each $U \in \mathcal{U}$ there is some $V \in \mathcal{V}$ with $U \subseteq V$.

In [10,3], the concepts of monotonically countably metacompact and monotonically metacompact were introduced. A space X is *monotonically* (*countably*) *metacompact* if there is a function r that associates with each (countable) open cover $\mathcal{U}$ of X an open point-finite refinement $r(\mathcal{U})$ that covers X, where r has the property that if $\mathcal{U}$ and $\mathcal{V}$ are open covers with $\mathcal{U} \prec \mathcal{V}$ then $r(\mathcal{U}) \prec r(\mathcal{V})$ (cf. [10,3]). In [3], it was proved that any metacompact Moore space is monotonically metacompact. In [3], it was also proved that if $(X, \tau, <)$ is a GO-space that is monotonically countably metacompact then (X, τ) is hereditarily paracompact, and it was remarked that a monotonically normal space that is monotonically countably metacompact must be paracompact.

In this note, we show that a monotonically normal space that is monotonically countably metacompact must be hereditarily paracompact. This answers a question which appears in Question 4.14 in [3]. We also point out that there is a gap in Proposition 3.8 which appears in [3]. Finally, we give a detailed proof of how to overcome the gap.

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In [7], G. Gruenhage showed that any compact monotonically Lindelöf T_2-space is first countable. In [6], it was proved that a monotonically meta-Lindelöf compact linearly ordered topological space (LOTS) X is first countable. A space X is *monotonically meta-Lindelöf* if each open cover $\mathcal{U}$ of X has a point-countable open refinement $r(\mathcal{U})$ such that if $\mathcal{U}$ and $\mathcal{V}$ are open covers with $\mathcal{U} \prec \mathcal{V}$ then $r(\mathcal{U}) \prec r(\mathcal{V})$ (cf. [6]). In this note we show that any compact monotonically meta-Lindelöf T_2-space is first countable. Thus some known conclusions are generalized.

All the spaces in this note are assumed to be T_1-spaces. The set of all positive integers is denoted by $\mathbb{N}$ and ω is $\mathbb{N} \cup \{0\}$. In notation and terminology we will follow [5,9].

2. Main results

A space X is *monotonically normal* if there is a function G which assigns to each ordered pair (H, K) of disjoint closed subsets of X an open set $G(H, K)$ such that:

(1) $H \subseteq G(H, K) \subseteq \overline{G(H, K)} \subseteq X \setminus K$;
(2) if (H', K') is a pair of disjoint closed subsets having $H \subseteq H'$ and $K \supseteq K'$, then $G(H, K) \subseteq G(H', K')$ (cf. [8]).

Lemma 1. *(Cf. [4, Corollary 1.3].) Monotone normality is hereditary.*

Lemma 2.2 in [8] also makes it clear that monotone normality is a hereditary property.

Lemma 2. *(Cf. [1, Theorem I].) A monotonically normal space is paracompact if and only if it does not have a closed subspace which is homeomorphic to a stationary subset of a regular uncountable cardinal.*

In [3], it was proved that if $(X, \tau, <)$ is a GO-space which is monotonically countably metacompact then (X, τ) is hereditarily paracompact. In fact, we have the following conclusion.

Theorem 3. *Suppose X is a monotonically normal space that is monotonically countably metacompact. Then X is hereditarily paracompact.*

Proof. Suppose X is not hereditarily paracompact. Then there is a subspace Y of X such that Y is not paracompact. By Lemma 1, we know that the space Y is monotonically normal. Thus by Lemma 2, we know that there is a closed subspace S of Y, which is homeomorphic to a stationary subset of a regular uncountable cardinal κ. We can assume that S is a subspace of κ. Let S^* be the set of limit points of S in X that belong to S. Thus $S^* \subseteq S$ and S^* is also a stationary subset of κ. If $\alpha \in S^*$, then we let α^+ be the first element of S^* that lies above α.

Let r be a monotone countable metacompactness operator for X. For each $\alpha \in S^*$, the family $\mathcal{U}_\alpha = \{X \setminus \{\alpha\}, X \setminus \{\alpha^+\}\}$ is an open cover of X. Thus $r(\mathcal{U}_\alpha) \prec \mathcal{U}_\alpha$ and there is some $V_\alpha \in r(\mathcal{U}_\alpha)$ such that $\alpha \in V_\alpha \subseteq X \setminus \{\alpha^+\}$. So there is some $f(\alpha) < \alpha$ such that $f(\alpha) \in S$ and $[f(\alpha), \alpha] \cap S \subseteq V_\alpha$, where $[f(\alpha), \alpha] = \{\beta\colon \beta \in S \text{ and } f(\alpha) \leqslant \beta \leqslant \alpha\}$. The set S^* is stationary in κ, then the Pressing Down Lemma provides some $\beta \in S$ such that the set $T = \{\alpha\colon \alpha \in S^* \text{ and } f(\alpha) = \beta\}$ is stationary in κ.

We can choose a strictly increasing sequence $\{\alpha_n\colon n \in \mathbb{N}\} \subset T$ such that $\alpha_n^+ < \alpha_{n+1}$ for each $n \in \mathbb{N}$. If $\mathcal{U} = \bigcup\{\mathcal{U}_{\alpha_n}\colon n \in \mathbb{N}\}$, then $\mathcal{U}_{\alpha_n} \prec \mathcal{U}$ for each $n \in \mathbb{N}$.

We let $A_1 = \{\alpha_n\colon n \in \mathbb{N}\}$ and $A_2 = \{\alpha_n, \alpha_n^+\colon n \in \mathbb{N}\}$. If $j_1 = 1$, then $[\beta, \alpha_{j_1}] \cap S \subseteq V_{\alpha_{j_1}}$. Since $r(\mathcal{U}_{\alpha_{j_1}}) \prec r(\mathcal{U})$, there is some $U_{\alpha_{j_1}} \in r(\mathcal{U})$ such that $V_{\alpha_{j_1}} \subseteq U_{\alpha_{j_1}} \subseteq X \setminus \{\delta_{j_1}\}$, where $\delta_{j_1} \in A_2$ and $\delta_{j_1} > \alpha_{j_1}$. Let $j_2 \in \mathbb{N}$ such that $j_2 > j_1$ and $\alpha_{j_2} > \delta_{j_1}$, where $\alpha_{j_2} \in A_1$. Thus $[\beta, \alpha_{j_2}] \cap S \subseteq V_{\alpha_{j_2}}$, where $V_{\alpha_{j_2}} \in r(\mathcal{U}_{\alpha_{j_2}})$. Thus there is some $U_{\alpha_{j_2}} \in r(\mathcal{U})$ such that $V_{\alpha_{j_2}} \subseteq U_{\alpha_{j_2}}$. So there is some $\delta_{j_2} \in A_2$ such that $U_{\alpha_{j_2}} \subseteq X \setminus \{\delta_{j_2}\}$. Since $\delta_{j_2} \notin V_{\alpha_{j_2}}$ and $[\beta, \alpha_{j_2}] \cap S \subseteq V_{\alpha_{j_2}} \subseteq U_{\alpha_{j_2}}$, we have that $\delta_{j_2} > \alpha_{j_2}$.

Suppose we have $\alpha_{j_i}, \delta_{j_i}, i \leqslant n$, such that $[\beta, \alpha_{j_i}] \cap S \subseteq V_{\alpha_{j_i}} \subseteq U_{\alpha_{j_i}} \subseteq X \setminus \{\delta_{j_i}\}$, where $V_{\alpha_{j_i}} \in r(\mathcal{U}_{\alpha_{j_i}})$, $U_{\alpha_{j_i}} \in r(\mathcal{U})$, $\delta_{j_i} \in A_2$, $\delta_{j_i} > \alpha_{j_i}$, and $\alpha_{j_{i+1}} > \delta_{j_i} > \alpha_{j_i}$ $(i+1 \leqslant n)$. Thus $U_{\alpha_{j_i}} \neq U_{\alpha_{j_k}}$ if $i, k \leqslant n$ and $i \neq k$.

We can choose $\alpha_{j_{n+1}} > \delta_{j_n}$ and $\alpha_{j_{n+1}} \in A_1$. Thus there is some $V_{\alpha_{j_{n+1}}} \in r(\mathcal{U}_{\alpha_{j_{n+1}}})$ such that $[\beta, \alpha_{j_{n+1}}] \cap S \subseteq V_{\alpha_{j_{n+1}}}$. Since $r(\mathcal{U}_{\alpha_{j_{n+1}}}) \prec r(\mathcal{U})$, there is some $U_{\alpha_{j_{n+1}}} \in r(\mathcal{U})$ such that $V_{\alpha_{j_{n+1}}} \subseteq U_{\alpha_{j_{n+1}}}$ and $U_{\alpha_{j_{n+1}}} \subseteq X \setminus \{\delta_{j_{n+1}}\}$, where $\delta_{j_{n+1}} \in A_2$. We know that $\delta_{j_{n+1}} > \alpha_{j_{n+1}}$, since $[\beta, \alpha_{j_{n+1}}] \cap S \subseteq V_{\alpha_{j_{n+1}}} \subseteq U_{\alpha_{j_{n+1}}} \subseteq X \setminus \{\delta_{j_{n+1}}\}$.

For each $i \leqslant n$, $\delta_{j_i} \in V_{\alpha_{j_{n+1}}} \subseteq U_{\alpha_{j_{n+1}}}$, and $U_{\alpha_{j_i}} \subseteq X \setminus \{\delta_{j_i}\}$, we know that $U_{\alpha_{j_{n+1}}} \neq U_{\alpha_{j_i}}$ for each $i \leqslant n$.

Thus we have $\beta \in U_{\alpha_{j_n}}$ for each $n \in \mathbb{N}$ and $U_{\alpha_{j_n}} \neq U_{\alpha_{j_m}}$ if $n, m \in \mathbb{N}$ and $n \neq m$. So $ord(\beta, r(\mathcal{U})) = |\{U\colon \beta \in U \text{ and } U \in r(\mathcal{U})\}| \geqslant \omega$. This is a contradiction with $ord(\beta, r(\mathcal{U})) < \omega$. So X is hereditarily paracompact. $\square$

Recall that every GO-space is monotonically normal. Thus we have the following corollary.

Corollary 4. *(Cf. [3, Proposition 3.4].) Suppose $(X, \tau, <)$ is a GO-space that is monotonically countably metacompact. Then (X, τ) is hereditarily paracompact.*

Theorem 5. *Suppose X is a monotonically normal space that is monotonically meta-Lindelöf. Then X is hereditarily paracompact.*

Proof. The proof is analogous to the proof of Theorem 3. Suppose there is a subspace $Y \subseteq X$, which is not paracompact. There is a closed subspace $S \subseteq Y$, which is homeomorphic to a stationary subset of a regular uncountable cardinal κ by Lemma 2. We assume $S \subseteq \kappa$.

Let S^* be the set of limit points of S in X that belong to S. Thus $S^* \subseteq S$ and S^* is also a stationary subset of κ. If $\alpha \in S^*$, then we let α^+ be the first element of S^* that lies above α.

Let r be a monotone meta-Lindelöf operator for X. For each $\alpha \in S^*$, the open cover $\mathcal{U}_\alpha = \{X \setminus \{\alpha\}, X \setminus \{\alpha^+\}\}$ has a point-countable open refinement $r(\mathcal{U}_\alpha)$. For each $\alpha \in S^*$, let $V_\alpha \in r(\mathcal{U}_\alpha)$ such that $\alpha \in V_\alpha \subseteq X \setminus \{\alpha^+\}$, and hence there is some $f(\alpha) < \alpha$ such that $f(\alpha) \in S$ and $[f(\alpha), \alpha] \cap S \subseteq V_\alpha$, where $[f(\alpha), \alpha] = \{\beta\colon \beta \in S \text{ and } f(\alpha) \leqslant \beta \leqslant \alpha\}$. By the Pressing Down Lemma, there is some $\beta \in S$ such that the set $T = \{\alpha\colon \alpha \in S^* \text{ and } f(\alpha) = \beta\}$ is stationary in κ.

Let $A = \{\alpha_\gamma\colon \gamma \in \omega_1\} \subseteq T$ such that $\alpha_{\gamma+1} > \alpha_\gamma^+$ for each $\gamma \in \omega_1$, and let $B = \{\alpha_\gamma, \alpha_\gamma^+\colon \gamma \in \omega_1\}$. If $\mathcal{U} = \bigcup\{\mathcal{U}_{\alpha_\gamma}\colon \gamma \in \omega_1\}$, then $r(\mathcal{U}_{\alpha_\gamma}) \prec r(\mathcal{U})$ for each $\gamma \in \omega_1$. Similar to the proof of Theorem 3, we can get $\{\alpha_{\gamma_q}\colon q \in \omega_1\} \subseteq A$ and $\{\delta_{\gamma_q}\colon q \in \omega_1\} \subseteq B$ such that $[\beta, \alpha_{\gamma_q}] \cap S \subseteq V_{\alpha_{\gamma_q}} \subseteq U_{\alpha_{\gamma_q}} \subseteq X \setminus \{\delta_{\gamma_q}\}$, where $U_{\alpha_{\gamma_q}} \in r(\mathcal{U})$, $\delta_{\gamma_q} > \alpha_{\gamma_q}$, and $\alpha_{\gamma_q} > \delta_{\gamma_p}$ for each $p < q$. Thus $U_{\alpha_{\gamma_p}} \neq U_{\alpha_{\gamma_q}}$ if $p, q \in \omega_1$ and $p \neq q$.

Thus $ord(\beta, r(\mathcal{U})) \geqslant \omega_1$. This is a contradiction with $ord(\beta, r(\mathcal{U})) \leqslant \omega$. Thus X is hereditarily paracompact. $\square$

Since the space ω_1 is not paracompact, we can get the following corollaries by Corollary 4 and Theorem 5.

Corollary 6. *(Cf. [10].) $\omega_1 + 1$ is not monotonically countably metacompact.*

Corollary 7. *(Cf. [2, Example 2.3].) $\omega_1 + 1$ is not monotonically Lindelöf.*

Corollary 8. *(Cf. [6, Proposition 2].) $\omega_1 + 1$ is not monotonically meta-Lindelöf.*

Lemma 9. *(Cf. [6, Proposition 4].) If $Y = X \cup \{p\}$ $(p \notin X)$ is the one-point compactification of the discrete space X of cardinally of ω_1, then the space Y is not monotonically meta-Lindelöf.*

In [7], it was proved that any compact monotonically Lindelöf T_2-space is first countable. By the following theorem, we can prove that any compact monotonically meta-Lindelöf T_2-space is first countable.

Theorem 10. *Suppose X is a regular monotonically meta-Lindelöf T_2-space. Let $Y \subseteq X$, and for each $y \in Y$, let $\phi(y)$ be an open neighborhood of y. Then there is an open neighborhood V_y of y such that $y \in V_y \subseteq \overline{V_y} \subseteq \phi(y)$ and satisfies that if $p \in \bigcap\{V_x\colon x \in Y'\}$ for some subset $Y' \subseteq Y$, then there is a countable subset $Y^* \subseteq Y'$ such that $Y' \subseteq \bigcup\{\phi(x)\colon x \in Y^*\}$.*

Proof. For each $x \in Y$, there is an open neighborhood O_x of x such that $x \in O_x \subseteq \overline{O_x} \subseteq \phi(x)$ by the regularity property of X. Thus $\mathcal{U}_x = \{\phi(x), X \setminus \overline{O_x}\}$ is an open cover of X. Let r be a monotone meta-Lindelöf operator for X. For each $x \in Y$, there is some $U_x \in r(\mathcal{U}_x)$ such that $x \in U_x \subseteq \phi(x)$. We let $V_x = O_x \cap U_x$.

Let $Y' \subseteq Y$ and let $p \in \bigcap\{V_x\colon x \in Y'\}$. Thus $\mathcal{U} = \bigcup\{\mathcal{U}_x\colon x \in Y'\}$ is an open cover of X and $\mathcal{U}_x \prec \mathcal{U}$ for each $x \in Y'$. Thus $r(\mathcal{U}_x) \prec r(\mathcal{U})$ for each $x \in Y'$, and hence there is some $M_x \in r(\mathcal{U})$ such that $U_x \subseteq M_x$. So there is some $y_x \in Y'$ such that $M_x \subseteq \phi(y_x)$ or $M_x \subseteq X \setminus \overline{O_{y_x}}$.

Suppose $M_x \subseteq X \setminus \overline{O_{y_x}}$. Then $V_{y_x} \cap M_x = \emptyset$, since $V_{y_x} \subseteq O_{y_x}$. This is a contradiction with $p \in V_{y_x}$. Thus $M_x \subseteq \phi(y_x)$ for each $x \in Y'$.

Since $p \in \bigcap\{V_x\colon x \in Y'\}$ and $V_x \subseteq M_x$ for each $x \in Y'$, we have $p \in \bigcap\{M_x\colon x \in Y'\}$. The family $\{M_x\colon x \in Y'\} \subseteq r(\mathcal{U})$ and $r(\mathcal{U})$ is point-countable, thus we let $\{M_x\colon x \in Y'\} = \{M_n\colon n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, there is some $y_n \in Y'$ such that $M_n \subseteq \phi(y_n)$. So $Y' \subseteq \bigcup\{\phi(y_n)\colon n \in \mathbb{N}\}$. If $Y^* = \{y_n\colon n \in \mathbb{N}\}$, then $Y' \subseteq \bigcup\{\phi(y)\colon y \in Y^*\}$. $\square$

The proof of the following theorem is analogous to the proof of Theorem 2.7 which appears in [7]. To assist the reader, we give the proof.

Theorem 11. *Compact monotonically meta-Lindelöf T_2-spaces are first countable.*

Proof. Suppose X is a compact monotonically meta-Lindelöf T_2-space but X is not first countable at a point $p \in X$. Then $\{p\}$ is not a G_δ-subset of X. We will define a decreasing sequence $\{H_\alpha\colon \alpha \in \omega_1\}$ of closed G_δ-sets containing p, and $x_\alpha \in H_\alpha$, satisfying:

(1) $x_\alpha \in H_\alpha \setminus H_{\alpha+1}$;
(2) if α is a limit ordinal and there is some $x \in H_\alpha \setminus \{p\}$ with $x \in \overline{\{x_\beta\colon \beta < \alpha\}}$, then $x_\alpha \in \overline{\{x_\beta\colon \beta < \alpha\}}$.

Let $H_0 = X$ and let $x_0 \in H_0 \setminus \{p\}$. Suppose H_β and x_β have been defined for all $\beta < \alpha$.

If $\alpha = \gamma + 1$ for some ordinal γ, then let K be any closed G_δ-set with $p \in K \subseteq X \setminus \{x_\gamma\}$, let $H_\alpha = K \cap H_\gamma$, and pick $x_\alpha \in H_\alpha \setminus \{p\}$. If α is a limit ordinal, then let $H_\alpha = \bigcap_{\beta<\alpha} H_\beta$. If there is $x \in H_\alpha \setminus \{p\}$ such that $x \in \overline{\{x_\beta\colon \beta < \alpha\}}$, then choose $x_\alpha \in H_\alpha \cap \overline{\{x_\beta\colon \beta < \alpha\}}$. Otherwise, let x_α be any point of $H_\alpha \setminus \{p\}$. Note that $H_\alpha \setminus \{p\} \neq \emptyset$ because $p \in H_\alpha$ and $\{p\}$ is not a G_δ-subset of X.

We have x_α and H_α defined for all $\alpha < \omega_1$. Let $\phi(x_\alpha) = X \setminus H_{\alpha+1}$ and for each $\alpha \in \omega_1$ let V_{x_α} be an open set with $x_\alpha \in V_{x_\alpha} \subseteq \overline{V_{x_\alpha}} \subseteq \phi(x_\alpha)$ as in Theorem 10.

Let $S = \{\alpha \in \omega_1\colon x_\alpha \in \overline{\{x_\beta\colon \beta < \alpha\}}\}$. We claim that S is not stationary. To obtain a contradiction, suppose S is stationary. For each $\alpha \in S$ there is some $g(\alpha) < \alpha$ with $x_{g(\alpha)} \in V_{x_\alpha}$ because V_{x_α} is a neighborhood of x_α and $x_\alpha \in \overline{\{x_\beta\colon \beta < \alpha\}}$. Then the Pressing Down Lemma gives a stationary set $M \subseteq S$ and an ordinal γ with $\gamma = g(\alpha)$ for each $\alpha \in M$. Therefore $x_\gamma \in \bigcap\{V_{x_\alpha}\colon \alpha \in M\}$ so that from Theorem 10 there is a countable set $M^* \subseteq M$ such that $\{x_\beta\colon \beta \in M\} \subseteq \bigcup\{\phi(x_\alpha)\colon \alpha \in M^*\}$. Since M^* is countable, there is some $\beta_0 \in M$ such that $\beta_0 > \alpha + 1$ for each $\alpha \in M^*$. Thus $x_{\beta_0} \in H_{\beta_0} \subseteq \bigcap\{H_{\alpha+1}\colon \alpha \in M^*\}$, and hence $x_{\beta_0} \notin X \setminus H_{\alpha+1} = \phi(x_\alpha)$ for all $\alpha \in M^*$. This contradiction shows that S cannot be stationary.

Because S is not stationary, there is a closed unbounded set $C \subseteq \omega_1$ with $C \cap S = \emptyset$. We claim that for each neighborhood U of p, we have that $\{x_\alpha\colon \alpha \in C\} \setminus U$ is finite. Suppose there is some open neighborhood U of p, such that $\{x_\alpha\colon \alpha \in C\} \setminus U$ is infinite. We let $A = \{x_{\alpha_n}\colon n \in \mathbb{N}\} \subseteq \{x_\alpha\colon \alpha \in C\} \setminus U$ and $x_{\alpha_n} \neq x_{\alpha_m}$ if $n \neq m$ and $n, m \in \mathbb{N}$. We assume $\alpha_n < \alpha_{n+1}$ for each $n \in \mathbb{N}$. Since $C \cap S = \emptyset$ and $x_n \in C$ for each $n \in \mathbb{N}$, the point $x_{\alpha_n} \notin \overline{\{x_\alpha\colon \alpha < \alpha_n\}}$. We also know that $x_{\alpha_n} \in H_{\alpha_n} \setminus H_{\alpha_n+1} \subseteq H_{\alpha_n} \setminus H_{\alpha_{n+1}}$ and $x_{\alpha_m} \in H_{\alpha_{n+1}}$ for each $m \geqslant n+1$. Thus $\{x_{\alpha_n}\colon n \in \mathbb{N}\}$ is a discrete subspace of X. Because X is compact, the set A has an accumulation point x in X. Since $x_{\alpha_n} \in H_{\alpha_n}$ and $H_{\alpha_{n+1}} \subset H_{\alpha_n}$ for each $n \in \mathbb{N}$, we know that $x \in \bigcap\{H_{\alpha_n}\colon n \in \mathbb{N}\}$. If $\gamma = \sup\{\alpha_n\colon n \in \mathbb{N}\}$, then $\bigcap\{H_{\alpha_n}\colon n \in \mathbb{N}\} = H_\gamma$ and hence $x \in H_\gamma$. Thus $x_\gamma \in \overline{\{x_\alpha\colon \alpha < \gamma\}}$.

On the other hand, $\alpha_n \in C$ for each $n \in \mathbb{N}$ and C is closed in ω_1, thus $\gamma \in C$. Since $C \cap S = \emptyset$, we have that $x_\gamma \notin \overline{\{x_\alpha\colon \alpha < \gamma\}}$. This contradiction shows that the set $\{x_\alpha\colon \alpha \in C\} \setminus U$ is finite for each neighborhood U of p.

Thus we have proved that $Y = \{x_\alpha\colon \alpha \in C\} \cup \{p\}$ is compact, and hence Y is closed in X. Since monotone meta-Lindelöfness is hereditary for closed subspaces (cf. [6, Proposition 3]), the space Y is monotonically meta-Lindelöf. For each $\alpha \in C$, we know that $x_\alpha \notin \overline{\{x_\beta\colon \beta < \alpha\}}$ because $C \cap S = \emptyset$. The set $H_{\alpha+1}$ is closed and $x_\alpha \notin H_{\alpha+1}$ for each $\alpha \in C$. Since $x_\beta \in H_{\alpha+1}$ for each $\beta > \alpha$, we have $x_\alpha \notin \overline{\{x_\beta\colon \beta \in C,\ \beta > \alpha\}}$. Thus the space $X_1 = \{x_\alpha\colon \alpha \in C\}$ is a discrete subspace of X and $|X_1| = \omega_1$. Since the space Y is a one-point compactification of X_1, we know that Y is not monotonically meta-Lindelöf by Lemma 9. This is a contradiction.

Thus X is first countable. $\square$

For any GO-space $(X, \tau, <)$. Let

$$I_\tau = \bigl\{x \in X\colon \{x\} \in \tau\bigr\};$$
$$R_\tau = \bigl\{x \in X \setminus I_\tau\colon [x, \rightarrow) \in \tau\bigr\};$$
$$L_\tau = \bigl\{x \in X \setminus I_\tau\colon (\leftarrow, x] \in \tau\bigr\}.$$

The following interesting conclusion appears in [3].

Theorem 12. *Let $(X, \tau, <)$ be a GO-space whose underlying LOTS $(X, \lambda, <)$ has a σ-closed-discrete dense subset. The following are equivalent:*

(1) (X, τ) *is monotonically metacompact*;
(2) (X, τ) *is monotonically countably metacompact*;
(3) *the set $R_\tau \cup L_\tau$ is σ-closed-discrete in (X, τ)*;
(4) *the set $R_\tau \cup L_\tau$ is σ-closed-discrete in (X, λ).*

In getting the above theorem, the following proposition was used in [3].

Proposition 13. *([3, Proposition 3.8]) Suppose $(X, \tau, <)$ is a GO-space for which the underlying LOTS $(X, \lambda, <)$ has a σ-closed-discrete dense subset. If (X, τ) is monotonically countably metacompact, then $R_\tau \cup L_\tau$ is σ-closed-discrete as a subspace of (X, τ) and as a subspace of (X, λ).*

The following statement was used in proving the previous proposition in [3].

Statement (*). Suppose $(X, \tau, <)$ is a GO-space for which the underlying LOTS $(X, \lambda, <)$ has a closed discrete subset D. If $p \in R_\tau \setminus D$ and there is some $y(p) > p$ such that $[p, y(p)) \cap D \neq \emptyset$, then we can decrease $y(p)$ if necessary, we may assume that $|[p, y(p)) \cap D| = 1$.

In fact, Statement (*) is not always true. The following is an example.

Example 14. Let S be the Sorgenfrey line and $X = S \setminus \{0\}$. Thus $(X, \tau, <)$ is a GO-space for which the underlying LOTS $(X, \lambda, <)$ has a closed discrete subset $D = \{\frac{1}{n}\colon n \in \mathbb{N}\}$. If $p = -1$, then $p \in R_\tau \setminus D$. Let $r(p) = 1$, thus we have $D \cap [p, r(p)) \neq \emptyset$. We cannot decrease $r(p)$ such that $|[p, r(p)) \cap D| = 1$.

Thus the previous Statement (*) is not always true. Since Statement (*) was used in the proof of Proposition 3.8 (i.e., Proposition 13 of this note) which appears in [3], Proposition 3.8 has a gap. Statement (*) was also used in the proof of Lemma 2.4 in [3], so Lemma 2.4 in [3] should be remarked. The following lemma is analogous to Lemma 2.3 in [3].

Lemma 15. *Let E be a closed discrete subset of a LOTS $(X, \lambda, <)$ and let $S \subseteq X$. Suppose that for each $x \in S$ there is some $e(x) \in E$ with $x < e(x)$ and such that the collection $\mathcal{C} = \{[x, e(x)]\colon x \in S\}$ is pairwise disjoint. Then the collection $\mathcal{C}$ is discrete in (X, λ) and the set S is a closed discrete subset of (X, λ).*

Proof. Let $y \in X$ and let U be a convex neighborhood of y in (X, λ), which contains at most one point of E. We can see that the set U meets at most two members of $\mathcal{C}$. Thus the collection $\mathcal{C}$ is discrete in (X, λ) and the set S is a closed discrete subset of (X, λ). $\square$

Lemma 16. *(Cf. [3, Lemma 2.4].) Suppose $(X, \tau, <)$ is a GO-space for which the underlying LOTS $(X, \lambda, <)$ has a σ-closed-discrete dense subset $D = \bigcup\{D(n)\colon n \in \mathbb{N}\}$. Suppose $S \subseteq X$ is σ-relatively discrete in the subspace topology τ_S and no point of S is isolated in τ. Thus S is σ-closed-discrete in (X, λ) and hence in (X, τ).*

Proof. It is enough to prove the lemma in case S is relatively discrete in (X, τ). Since D is σ-closed-discrete in (X, λ), thus $S \cap D$ is σ-closed-discrete in (X, λ). In what follows, we just need to prove that $S \setminus D$ is σ-closed-discrete in (X, λ). We let $S' = S \setminus D$.

Let $E_\tau = X \setminus (I_\tau \cup R_\tau \cup L_\tau)$. We just need to prove that $S_1 = S' \cap (R_\tau \cup E_\tau)$ and $S_2 = S' \cap (L_\tau \cup E_\tau)$ are σ-closed-discrete in (X, λ), since $S \cap I_\tau = \emptyset$.

For each $x \in S_1$, let $U(x)$ be a convex τ-neighborhood of x with $U(x) \cap S = \{x\}$, since the set S is relatively discrete in the subspace topology τ_S. Thus there is some $p(x) > x$ such that $[x, p(x)) \subseteq U(x)$ and $(x, p(x)) \neq \emptyset$. Let $N(x)$ be the smallest such that $(x, p(x)) \cap D(N(x)) \neq \emptyset$ and let $d(x) \in (x, p(x)) \cap D(N(x))$. If $M_n = \{x\colon x \in S_1 \text{ and } N(x) = n\}$ for each $n \in \mathbb{N}$, then $S_1 = \bigcup\{M_n\colon n \in \mathbb{N}\}$. Let $n \in \mathbb{N}$ and for any distinct points $x_1, x_2 \in M_n$, we know that $[x_1, d(x_1)] \cap [x_2, d(x_2)] = \emptyset$ because $U(x_i) \cap S = \{x_i\}$ for each $i \in \{1, 2\}$.

If $\mathcal{F}_n = \{[x, d(x)]\colon x \in M_n\}$, then $\mathcal{F}_n$ is a closed discrete family in (X, λ) by Lemma 15. Thus M_n is closed discrete in (X, λ), and hence S_1 is σ-closed-discrete in (X, λ). Similarly, we can prove that S_2 is σ-closed-discrete in (X, λ). Thus S is σ-closed-discrete in (X, λ). $\square$

By the proof of Proposition 3.8 which appears in [3], we have the following lemma.

Lemma 17. *Suppose $(X, \tau, <)$ is a GO-space and $y_n \in R_\tau$, $y_{n+1} < y_n$ for each $n \in \mathbb{N}$. If (X, τ) is monotonically countably metacompact, r is a monotone countable metacompactness operator for (X, τ), and $U_n \in r(\mathcal{U}_n)$ such that $y_n \in U_n$, where $\mathcal{U}_n = \{(\leftarrow, y_n), [y_n, \rightarrow)\}$ for each $n \in \mathbb{N}$, then $\{U_n\colon n \in \mathbb{N}\}$ is point-finite.*

Proof. Suppose $\{U_n\colon n \in \mathbb{N}\}$ is not point-finite, we can assume $\bigcap\{U_n\colon n \in \mathbb{N}\} \neq \emptyset$ and $|\{U_n\colon n \in \mathbb{N}\}| = \omega$. Let $p \in \bigcap\{U_n\colon n \in \mathbb{N}\}$. Since $y_n \in U_n$, $U_n \in r(\mathcal{U}_n)$, and $\mathcal{U}_n = \{(\leftarrow, y_n), [y_n, \rightarrow)\}$, we know that $U_n \subseteq [y_n, \rightarrow)$. We have $y_n \leqslant p$ for each $n \in \mathbb{N}$, since $p \in \bigcap\{U_n\colon n \in \mathbb{N}\}$.

If $\mathcal{U} = \bigcup\{\mathcal{U}_n\colon n \in \mathbb{N}\}$, then $\mathcal{U}_n \prec \mathcal{U}$ and hence $r(\mathcal{U}_n) \prec r(\mathcal{U})$ for each $n \in \mathbb{N}$.

Since $y_n \leqslant p$ for each $n \in \mathbb{N}$, there is some $m_n \in \mathbb{N}$, $m_n \geqslant n$ such that $U_n \subseteq V_{m_n} \subseteq [y_{m_n}, \rightarrow)$, where $V_{m_n} \in r(\mathcal{U})$.

Let $j_1 = 1$ and $j_{n+1} = m_{j_n} + 1$ for each $n \in \mathbb{N}$. Thus $p \in U_{j_n} \subseteq V_{m_{j_n}} \subseteq [y_{m_{j_n}}, \rightarrow)$ for each $n \in \mathbb{N}$. Since $y_{j_{n+1}} < y_{m_{j_n}}$, we know that $y_{j_{n+1}} \notin V_{m_{j_n}}$. Thus $\{V_{m_{j_n}}\colon n \in \mathbb{N}\} \subseteq r(\mathcal{U})$ and $|\{V_{m_{j_n}}\colon n \in \mathbb{N}\}| = \omega$. This contradicts that $r(\mathcal{U})$ is point-finite. Thus $\{U_n\colon n \in \mathbb{N}\}$ is point-finite. $\square$

Similarly, we have:

Lemma 18. *Suppose $(X, \tau, <)$ is a GO-space and $y_n \in L_\tau$, $y_{n+1} > y_n$ for each $n \in \mathbb{N}$. If (X, τ) is monotonically countably metacompact, r is a monotone countable metacompactness operator for (X, τ), and $U_n \in r(\mathcal{U}_n)$ such that $y_n \in U_n$, where $\mathcal{U}_n = \{(\leftarrow, y_n], (y_n, \rightarrow)\}$ for each $n \in \mathbb{N}$, then $\{U_n\colon n \in \mathbb{N}\}$ is point-finite.*

The following is the proof of Proposition 13.

Proof of Proposition 13. Lemma 16 guarantees that R_τ is σ-closed-discrete in (X, λ) if and only if R_τ is σ-relatively discrete in (X, τ). The same assertion holds for L_τ.

Let $D = \bigcup\{D(n)\colon n \in \mathbb{N}\}$ be a σ-closed-discrete dense subset in the underlying LOTS (X, λ). Let r be a monotone countable metacompactness operator for (X, τ). Because $R_\tau \cap D$ is σ-closed-discrete in (X, λ), it is enough to show that the set $R' = R_\tau \setminus D$ is σ-closed-discrete in (X, λ). By Lemma 16, it is enough to show that the set R' is σ-relatively discrete in (X, τ). In what follows, we show that the set R' is σ-relatively discrete in (X, τ).

For each $p \in R'$, let $\mathcal{U}(p) = \{(\leftarrow, p), [p, \rightarrow)\}$ and find $r(\mathcal{U}(p))$. Choose any $O(p) \in r(\mathcal{U}(p))$ that contains p and note that $O(p) \subseteq [p, \rightarrow)$ because $r(\mathcal{U}(p)) \prec \mathcal{U}(p)$.

For each $p \in R'$ and for each $n \in \mathbb{N}$, the set $D(n)$ is a closed discrete set in (X, λ), $p \notin D_n$, and $p \notin I_\tau$. Thus there is some $y_n(p) \in X$ such that $(p, y_n(p)) \cap D_n = \emptyset$. Since $p \notin I_\tau$ and $[p, y_n(p))$ is open in (X, τ), the set $(p, y_n(p)) \neq \emptyset$. We can assume $y_{n+1}(p) < y_n(p)$ for each $n \in \mathbb{N}$. Since the set $D = \bigcup\{D(n)\colon n \in \mathbb{N}\}$ is dense in (X, λ), we know that the sequence $\{y_n(p)\}_{n\in\mathbb{N}}$ converges to the point p in (X, λ). Thus the point p has a countable open neighborhood base in (X, τ). Since the set O_p is an open neighborhood of the point p in (X, τ), there is some $n(p) \in \mathbb{N}$ such that $[p, y_{n(p)}(p)) \subseteq O_p$. Thus there is some $m(p) \in \mathbb{N}$ such that $(p, y_{n(p)}(p)) \cap D(m(p)) \neq \emptyset$. If $R'_m = \{p\colon p \in R'$ such that $m(p) = m\}$, then $R' = \bigcup\{R'_m\colon m \in \mathbb{N}\}$. In what follows, we show that R'_m is a relatively discrete subspace in (X, τ).

Suppose there is some $m \in \mathbb{N}$ such that R'_m is not a relatively discrete subspace in (X, τ). Then there is some $p \in R'_m$ and a sequence $\{p_k\colon k \in \mathbb{N}\} \subseteq R'_m \setminus \{p\}$ which converges to the point p in (X, τ) by the first countable property of the point p in (X, τ). We can assume that $p_{k+1} < p_k$ for each $k \in \mathbb{N}$.

For each $k \in \mathbb{N}$, $[p_k, y_{n(p_k)}(p_k)) \subseteq O(p_k)$ and $(p_k, y_{n(p_k)}(p_k)) \cap D(m) \neq \emptyset$. Suppose for each $k \in \mathbb{N}$, there is some $l_k \in \mathbb{N}$, $l_k > k$, such that $[p_{l_k+1}, p_{l_k}) \cap D(m) \neq \emptyset$. We let $a_{l_k} \in [p_{l_k+1}, p_{l_k}) \cap D(m)$. We can assume $l_{k+1} > l_k$. Thus the sequence $\{a_{l_k}\}_{k\in\mathbb{N}}$ converges to the point p in (X, λ). This contradicts that $D(m)$ is closed discrete in (X, λ). Thus there is some $k_0 \in \mathbb{N}$ such that $[p_{k+1}, p_k) \cap D(m) = \emptyset$ for each $k \geqslant k_0$. Since $p_{k+1} \in R'_m$ for each $k \geqslant k_0$, we know that $[p_{k+1}, y_{n(p_{k+1})}(p_{k+1})) \cap D(m) \neq \emptyset$ and $p_{k+1} \notin D(m)$. Thus $p_{k_0} \in [p_{k+1}, y_{n(p_{k+1})}(p_{k+1}))$ for each $k > k_0$. Thus $p_{k_0} \in O(p_{k+1})$ for each $k > k_0$. Since $O(p_{k+1}) \subseteq [p_{k+1}, \rightarrow)$ and the sequence $\{p_{k+1}\colon k > k_0\}$ is decreasing, the family $\{O(p_{k+1})\colon k > k_0\}$ is not point-finite. By Lemma 17, we have that $\{O(p_{k+1})\colon k > k_0\}$ is point-finite. This contradiction shows that the set R'_m is a relatively discrete subspace in (X, τ) for each $m \in \mathbb{N}$. Thus R' is a σ-relatively discrete subspace in (X, τ).

Thus R' is a σ-closed-discrete subspace in (X, λ) by Lemma 16. So R_τ is σ-closed-discrete in (X, λ).

By similar argument and Lemma 18, we can show that L_τ is σ-closed-discrete in (X, λ). Thus $R_\tau \cup L_\tau$ is σ-closed-discrete as a subspace of (X, τ) and as a subspace of (X, λ). □

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